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Error and Sensitivity Analysis Scheme of a New Data Compression Technique in Estimation

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A new data compression technique of a new class was recently introduced. Although the rationale and the equations involved in the application of the technique were fully presented, an error analysis of the actual (true) estimation error was not provided. Due to the unconventional operations involved in the application of the technique, the standard error and sensitivity analysis routines could not be utilized. This paper presents a genuine error and sensitivity analysis scheme which handles the error analysis problem of the new data compression technique. The scheme is based on a reformulation of the data compression equations in a compact form. This, together with the proper sequencing of the operations involved in the execution of the data compression technique, makes the technique compatible with the standard practice of error and sensitivity analysis. Particular care is given to the analysis of biased estimates. An example involving data compression in an inertial navigation system is introduced, and computer results are presented and discussed.

Nomenclature

C_y	= second moment matrix of y
G_y	= weighting matrix of the process noise of y
H	= observation matrix
K	= gain matrix
k	= kth time instant, $t(k)$
N	= number of measurements between successive renovation instants
P	= assumed covariance matrix
p	= pth renovation instant, $t(Np)$
Q	= covariance matrix of the process noise
R	= covariance matrix of the measurement noise
T	= transformation matrix
$t(j)$	= jth time instant
V	= true covariance matrix
v	= measurement noise vector
w	= process noise vector
x	= state vector
y	= augmented state vector
z	= observation vector
Γ	= adjustment matrix
$\Delta[\cdot]$	= matrix difference due to modeling error
μ	= mean vector
$\Phi(k,k-1)$	= transfer matrix from $t(k-1)$ to $t(k)$
$[\cdot]_f$	= full-order vector/matrix
$[\cdot](j)$	= value of $[\cdot]$ at the jth instant
$[\cdot](j\|i)$	= value of $[\cdot](j)$ conditioned on i observations
$[\cdot]_r$	= reduced-order vector/matrix
$[\cdot]^T$	= transpose of a general matrix $[\cdot]$
$[\cdot]_t$	= truth model vector/matrix
$[\cdot]_y$	= augmented system matrix
${}^{\eta}[\cdot]$	= subvector of order η
${}^{\eta\eta}[\cdot]$	= submatrix of order $\eta\times\eta$
${}^{\nu}[\cdot]$	= subvector of order ν
${}^{\nu\eta}[\cdot]$	= submatrix of order $\nu\times\eta$
${}^{\nu\nu}[\cdot]$	= submatrix of order $\nu\times\nu$
$[\hat{\cdot}]$	= estimated value of $[\cdot]$
$[\tilde{\cdot}]$	= estimation error of $[\cdot]$
$[\cdot]^{-1}$	= inverse of a square matrix $[\cdot]$
FO	= full-order
FOM	= full-order model
RO	= reduced-order
ROM	= reduced-order model

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I. Introduction

A RECOGNIZED shortcoming of the Kalman filter state estimator is the relative complexity of the computation needed to determine its optimal gain. As this gain has to be computed for each data point, the computation load sets an upper bound on the rate at which measured data can be processed. When new measurement data are obtained at a high rate, a situation may arise in which the filter is incapable of processing all of the available data and some of it has to be discarded. Thus, the estimator is not using all of the information contained in the acquired data.

Data compression, or preprocessing techniques, are aimed at alleviating this problem. Several methods for data compression have previously been suggested. They are divided into three classes. The first class of techniques is based on averaging schemes. The algorithms outlined by Schmidt,[1] Joglekar,[2] and Bar-Shalom[3] belong to this class which contains most of the techniques presented in the literature. Dressler and Ross[4] presented a scheme which belongs to the second class. According to this data compression method, the filter gain is computed in a piecewise constant manner. After each gain computation, it is held constant until its next computation and is used to process measurement data which are obtained between the two gain computation instants. Thus, the gain is computed at a rate smaller than that at which the data are acquired.

Recently, a method belonging to a new class of data compression techniques was introduced.[5,6] According to this technique, a reduced-order Kalman filter is propagated and updated using all of the measured data. At a slower rate, this reduced-order filter renovates a full-order filter; consequently, most of the computation load is that of a reduced-order filter gain, a task which the computer can easily and frequently perform. The reduced-order filter yields an estimate that is treated as the estimate of a part of the full-order state vector. It also yields a covariance matrix which is treated as a part of the full-order filter covariance matrix. Obviously, this technique is a suboptimal one on two accounts:

1) A reduced-order model is used to compute a part of the full-order filter (which is then used to renovate the rest of the full-order filter).

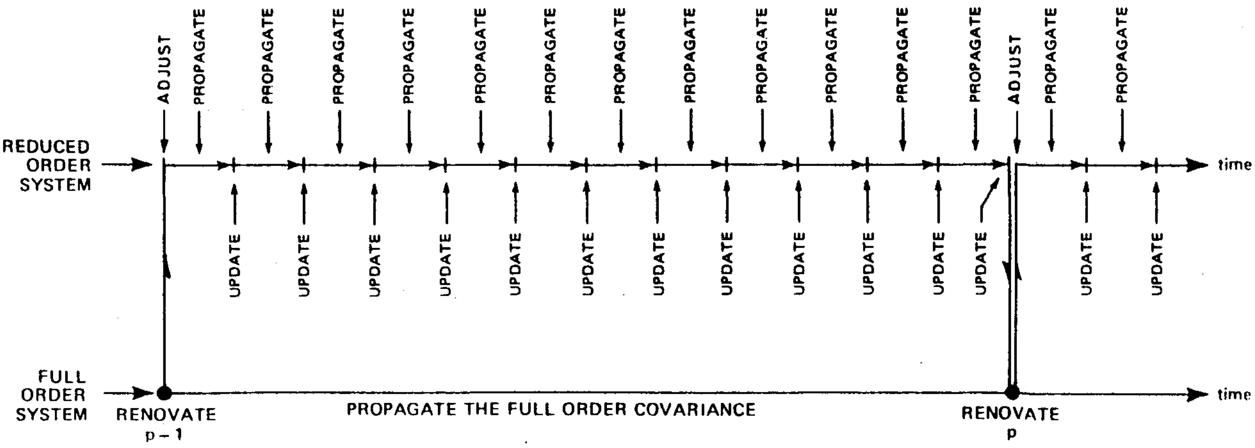

Fig. 1 Time sequence of the multirate data compression technique.

2) The algorithm used to renovate the rest of the full-order filter is exact only when one measurement is processed at each renovation instant.

As this is a suboptimal procedure, a true error analysis has to be carried out in order to evaluate the proposed algorithm. It should be noted that Refs. 5 and 6 only introduce and develop the new data compression technique; therefore, such an analysis is inevitable.

The standard sensitivity and error analysis methods of suboptimal filters[7-9] are inadequate in this case, since they are incapable of handling a multirate filter such as this one and they cannot handle filters which undergo operations (i.e., renovation) other than conventional updates. Therefore, a new error and sensitivity analysis scheme has been developed to analyze the data compression technique presented in Refs. 5 and 6. This paper describes the new scheme. Section II outlines the analyzed data compression technique which is reformulated compactly in Sec. III. The error and sensitivity scheme itself is presented in Sec. IV. An example of the application of the scheme to an inertial navigation system (INS) is discussed in Sec. V. The conclusions of this work follow in Sec. VI.

II. Outline of the Analyzed Data Compression Technique

In order to explain the error analysis scheme, it is useful to sketch in more detail the data compression technique[5,6] to which the analysis is applied. According to this data compression technique, a reduced-order filter, whose discrete-time state at time $t(k)$ is denoted by $x_r(k)$, is propagated and (Kalman) updated at every measurement instant. The update yields $\hat{x}_r(k|k)$, the estimate of $x_r(k)$. Every N measurements (and updates) a part, ${}^{\nu}\hat{x}_r(k|k)$, of $\hat{x}_r(k|k)$ is used at the pth renovation instant $[t(Np)=t(k)]$ to renovate[5,6] the full-order estimator (see Fig. 1).

The vector ${}^{\nu}\hat{x}_r(k|k)$ contains the first ν elements of $\hat{x}_r(k|k)$, where these elements are the states involved in the measurement equation. The update of the state of the full-order estimator $\hat{x}_f$ is done as follows. The first ν states of the full-order estimator are replaced by ${}^{\nu}\hat{x}_r(k|k)$, consequently

$$ {}^{\nu}\hat{x}_f(Np|Np) = {}^{\nu}\hat{x}_r(k|k) \tag{1} $$

The rest of the state, ${}^{\eta}\hat{x}_f(Np|Np)$, of the full-order estimator is *renovated* using

$$ \begin{aligned} {}^{\eta}\hat{x}_f(Np|Np) &= {}^{\eta}\hat{x}_f(Np|Np-N) \\ &+ {}^{\nu\eta}P_f^T(Np|Np-N)\,{}^{\nu\nu}P_f^{-1}(Np|Np-N) \\ &\cdot[{}^{\nu}\hat{x}_f(Np|Np) - {}^{\nu}\hat{x}_f(Np|Np-N)] \end{aligned} \tag{2} $$

where the vector $\hat{x}_f(Np|Np-N)$ is obtained by projecting the full-order state estimate, $\hat{x}_f(Np-N|Np-N)$, from the previous renovation instant (N measurements ago). The matrices ${}^{\nu\eta}P_f(Np|Np-N)$ and ${}^{\nu\nu}P_f(Np|Np-N)$ needed in Eq. (2) are the following submatrices of $P_f(Np|Np-N)$, whose computation will be discussed shortly,

$$ P_f(Np|Np-N) = \left[\begin{array}{c|c} {}^{\nu\nu}P_f(Np|Np-N) & {}^{\nu\eta}P_f(Np|Np-N) \\ \hline {}^{\nu\eta}P_f^T(Np|Np-N) & {}^{\eta\eta}P_f(Np|Np-N) \end{array}\right] \tag{3} $$

The matrix $P_f(Np|Np-N)$ is obtained by propagating $P_f(Np-N|Np-N)$ from the previous renovation instant (N measurements ago). After $\hat{x}_f(Np|Np)$ is computed using renovation, $P_f(Np|Np-N)$ is also renovated and propagated to the next renovation point. The renovation of $P_f(Np|Np-N)$ is done as follows. Let ${}^{\nu\nu}P_r(k|k)$ be the upper left submatrix of the filter-assumed covariance matrix of the error of the reduced-order state estimate ${}^{\nu}\hat{x}_r(k|k)$ at the renovation instant. Define G and D as follows:

$$ G \triangleq {}^{\nu\nu}P_r(k|k)\,{}^{\nu\nu}P_f^{-1}(Np|Np-N) \tag{4a} $$

$$ D \triangleq {}^{\nu\nu}P_f^{-1}(Np|Np-N)(I-G) \tag{4b} $$

then

$$ P_f(Np|Np) = \tag{5} $$

$$ \left[\begin{array}{c|c} {}^{\nu\nu}P_r(k|k) & G\cdot{}^{\nu\eta}P_f(Np|Np-N) \\ \hline {}^{\nu\eta}P_f^T(Np|Np-N)G^T & \begin{array}{l} {}^{\eta\eta}P_f(Np|Np-N) - {}^{\nu\eta}P_f^T(Np|Np-N) \\ \cdot D\cdot{}^{\nu\eta}P_f(Np|Np-N) \end{array} \end{array}\right] $$

The operations described in Eqs. (1) and (2) are called *state renovation* and those in Eqs. (4) and (5) are called *covariance renovation*.[5,6] After completing the renovation stage, the reduced-order filter is adjusted as follows. The state estimates not included in the first ν states are replaced by the corresponding values in $\hat{x}_f(Np|Np)$. The elements of the filter-assumed covariance matrix, which are not included in ${}^{\nu\nu}P_r(k|k)$, are replaced by the corresponding elements in $P_f(Np|Np)$. These replacement procedures are called reduced-order filter *adjustment*. The time sequence of the operations involved in this data compression technique are illustrated in Fig. 1. For further details concerning this algorithm, the reader is referred to Refs. 5 and 6.

III. Compact Algorithm for Renovation and Adjustment

The data compression algorithm described in the preceding section is not compatible with known error and sensitivity analysis routines.[7-9] The reasons for this are that these routines cannot handle a multirate combination of filter/estimator such as this one and, in addition, they are not capable of handling filters which undergo unconventional operations such as renovation.

This shortcoming of the known error and sensitivity analysis routines is successfully overcome in this work by reformulating the renovation and adjustment procedures in a compact form suitable for the standard error and sensitivity analysis routines. This form is derived in this section. Rewrite Eqs. (1) and (2) in the following form:

$$^{\nu}\hat{x}_f(Np\,|Np) = {}^{\nu}\hat{x}_f(Np\,|Np-N) + {}^{\nu\nu}I[{}^{\nu}\hat{x}_r(k\,|k) - {}^{\nu}\hat{x}_f(Np\,|Np-N)] \tag{6a}$$

$$^{\eta}\hat{x}_f(Np\,|Np) = {}^{\eta}\hat{x}_f(Np\,|Np-N) + {}^{\nu\eta}P_f^T(Np\,|Np-N)\,{}^{\nu\nu}P_f^{-1}(Np\,|Np-N) \cdot [{}^{\nu}\hat{x}_r(k\,|k) - {}^{\nu}\hat{x}_f(Np\,|Np-N)] \tag{6b}$$

where $^{\nu\nu}I$ is an $\nu \times \nu$ identity matrix. These equations can be combined into one equation:

$$\hat{x}_f(Np\,|Np) = \hat{x}_f(Np\,|Np-N) + K_f(Np)\,[{}^{\nu}x_r(k\,|k) - H_f(Np)\hat{x}_f(Np\,|Np-N)] \tag{7}$$

where

$$K_f(Np) = \left[\begin{array}{c} ^{\nu\nu}I \\ \hline ^{\nu\eta}P_f^T(Np\,|Np-N)\,{}^{\nu\nu}P_f^{-1}(Np\,|Np-N) \end{array}\right] \tag{8}$$

and

$$H_f(Np) = [\,{}^{\nu\nu}I \,|\, 0\,] \tag{9}$$

Note the resemblance between Eq. (7) and the standard Kalman state estimator. It is seen that $^{\nu}\hat{x}_r(k\,|k)$ corresponds to the measurement vector in the standard Kalman filter, although $^{\nu}\hat{x}_r(k\,|k)$ is actually a filtered measurement vector. Now, the renovated $P_f(Np\,|Np)$ of Eq. (5) can be rewritten compactly as stated in the following proposition.

Proposition 1:

$$P_f(Np\,|Np) = [I - K_f(Np)H_f(Np)]P_f(Np\,|Np-N)\,[I - K_f(Np)H_f(Np)]^T + K_f(Np)\,{}^{\nu\nu}P_r(k\,|k)K_f^T(Np) \tag{10}$$

Proof: Use Eqs. (8) and (9) to obtain

$$K_f(Np)\,{}^{\nu\nu}P_r(k\,|k)K_f^T(Np)$$

$$= \left[\begin{array}{c|c} ^{\nu\nu}P_r(k\,|k) & ^{\nu\nu}P_r(k\,|k)\,{}^{\nu\nu}P_f^{-1}(Np\,|Np-N)\,{}^{\nu\eta}P_f(Np\,|Np-N) \\ \hline ^{\nu\eta}P_f^T(Np\,|Np-N)\,{}^{\nu\nu}P_f^{-1}(Np\,|Np-N) \cdot {}^{\nu\nu}P_r(k\,|k) & ^{\nu\eta}P_f^T(Np\,|NP-N)\,{}^{\nu\nu}P_f^{-1}(Np\,|Np-N)\,{}^{\nu\nu}P_r(k\,|k) \cdot {}^{\nu\nu}P_f^{-1}(Np\,|Np-N)\,{}^{\nu\eta}P_f(Np\,|Np-N) \end{array}\right] \tag{11}$$

and

$$[I - K_f(Np)H_f(Np)]P_f(Np\,|Np-N)\,[I - K_f(Np)H_f(Np)]^T$$

$$= \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & ^{\eta\eta}P_f(Np\,|Np-N) - {}^{\nu\eta}P_f^T(Np\,|Np-N) \cdot {}^{\nu\nu}P_f^{-1}(Np\,|Np-N) \cdot {}^{\nu\eta}P_f(Np\,|Np-N) \end{array}\right] \tag{12}$$

Now use Eq. (4) to show that the sum of the right-hand side of Eqs. (11) and (12) is equal to the right-hand side of Eq. (5), which ends the proof ∴.

Note the resemblance between the renovated $P_f(Np\,|Np)$ given in Eq. (10) and the well-known Joseph's algorithm[10] for the Kalman update of the covariance matrix.

From the outline of the data compression technique in Sec. II it is clear that the reduced-order state x_r is a subvector of the full-order state x_f, that is,

$$x_r = \Gamma x_f \tag{13}$$

Then the adjustment of the reduced-order filter, defined in Sec. II, can be expressed as follows:

$$\hat{x}_r(Np\,|Np) = \Gamma\hat{x}_f(Np\,|Np) \tag{14a}$$

$$P_r(Np\,|Np) = \Gamma P_f(Np\,|Np)\Gamma^T \tag{14b}$$

Note the similarity between the adjustment procedure as expressed in Eqs. (14) and the instantaneous state reset control employed when conventional state estimators are used.

Equations (7), (10), and (14) together with Eqs. (8), (9), and (13) cast the renovation and adjustment procedures involved in the new data compression technique into a form suitable for the application of the standard error and sensitivity analysis routines.

IV. The Error and Sensitivity Analysis Scheme

Following the description of the data compression technique which is to be analyzed, it is clear that one has to consider three system models. The first model is that of the real world. This model describes the behavior of the actual process, or system, to the best of one's knowledge and ability. This model will be called the *truth model*, its state vector in the discrete formulation at time $t(k)$ will be denoted by $x_t(k)$. The second model is that assumed by the full-order (FO) filter. This model will be called the *FO design model*. Its discrete state vector, at time $t(k)$, is denoted by $x_f(k)$. The FO filter is designed to yield the estimate, $\hat{x}_f(Np\,|Np)$, of the state $x_f(k)$ of the real world at the renovation instant where $k = Np$. Finally, the third model, implied by the data compression technique, is that of the reduced-order (RO) filter. This model will be called the *RO design model*. Its state vector at time $t(k)$ will be denoted by $x_r(k)$ and its estimate by $\hat{x}_r(k|k)$. Its first ν elements are used in the state renovation process of $\hat{x}_f(Np|Np)$ as indicated by Eq. (7). The upper $\nu \times \nu$ covariance submatrix of $x_r(k) - \hat{x}_r(k|k)$, which was denoted by ${}^{\nu\nu}P_r(k|k)$, is used in the renovation of the matrix $P_f(Np|Np)$ given by Eq. (10). The exact description of these three models and the interrelations in the error and sensitivity analysis scheme is described next.

A. The Truth Model

It is assumed (to the best of our knowledge) that the actual process, whose state is to be estimated, satisfies the discrete-state equation

$$x_t(k) = \Phi_t(k,k-1)x_t(k-1) + w_t(k-1) \tag{15}$$

where $\Phi_t(k,k-1)$ is the state transition matrix from time $t(k-1)$ to $t(k)$ and $\{w_t(k-1)\}$ is a random, zero-mean, white noise sequence, that is

$$E\{w_t(k)\} = 0 \qquad \forall k \geq 0 \tag{16a}$$

$$E\{w_t(k)w_t^T(j)\} = Q_t(k)\delta(k-j) \qquad \forall k,j \geq 0 \tag{16b}$$

where $\delta(k-j)$ is the Kronecker delta function and E denotes expected value. The measurement (observation) model of the actual system at time $t(k)$ is

$$z(k) = H_t(k)x_t(k) + v_t(k) \tag{17}$$

where $H_t(k)$ is the observation matrix at $t(k)$ and $\{v_t(k)\}$ is a random, zero-mean, white noise sequence, that is,

$$E\{v_t(k)\} = 0 \qquad \forall k \geq 0 \tag{18a}$$

$$E\{v_t(k)v_t^T(j)\} = R_t(k)\delta(k-j) \qquad \forall k,j \geq 0 \tag{18b}$$

$\{v_t(k)\}$ and $\{w_t(k)\}$ are assumed to be uncorrelated.

The initial state $x_t(0)$ of the system is a random vector, with the following statistics:

$$E\{x_t(0)\} = \mu_t \tag{19a}$$

$$E\{x_t(0)x_t^T(0)\} = P_t(0) \tag{19b}$$

and is uncorrelated with the process and measurement white sequences, that is

$$E\{x_t(0)w_t^T(k)\} = 0 \qquad \forall k \geq 0 \tag{20a}$$

$$E\{x_t(0)v_t^T(k)\} = 0 \qquad \forall k \geq 0 \tag{20b}$$

B. The Full-Order Filter

The FO filter is mechanized to yield the best estimate of the state of the actual process. Unfortunately, implementation considerations and/or modeling uncertainties will lead us to choose a model somewhat different from the truth model discussed earlier.

1. The Full-Order Filter Design Model (FOM)

The chosen model is the FOM, whose discrete-time representation in the state space is given by

$$x_f(k) = \Phi_f(k,k-1)x_f(k-1) + w_f(k-1) \tag{21}$$

where $\Phi_f(k,k-1)$ is the state transition matrix from time $t(k-1)$ to time $t(k)$. $\{w_f(k-1)\}$ is the assumed random, zero-mean, process white noise sequence, that is,

$$E\{w_f(k)\} = 0 \qquad \forall k \geq 0 \tag{22a}$$

$$E\{w_f(k)w_f^T(j)\} = Q_f(k)\delta(k-j) \qquad \forall k,j \geq 0 \tag{22b}$$

The initial state $x_f(0)$ of the FOM is also a random vector with the following statistics:

$$E\{x_f(0)\} = \mu_f \tag{23a}$$

$$E\{x_f(0)x_f^T(0)\} = P_f(0) \tag{23b}$$

and is uncorrelated with the process white noise, that is,

$$E\{x_f(0)w_f^T(k)\} = 0 \qquad \forall k \geq 0 \tag{24}$$

Note that no observation model is incorporated into the FOM. This is implied by the renovation algorithm of the data compression technique.

2. The Full-Order Filter Equations

The estimate $\hat{x}_f$ generated by the FO filter is propagated between successive renovations in consistence with the FOM as follows:

$$\hat{x}_f(k|Np-N) = \Phi_f(k,k-1)\hat{x}_f(k-1|Np-N) \qquad \hat{x}_f(0|0) = \mu_f \tag{25}$$

At $k = Np$ the state renovation is performed according to Eq. (7). Generally, the FO filter will fail to correctly estimate the state of the truth model for two reasons: because of modeling differences and because of the inherent inaccuracy of the renovation algorithm.

C. The Reduced-Order Filter

As discussed earlier, the RO design model is chosen to reproduce the dynamic behavior of the actual process in relatively short time intervals. Obviously, this model will differ from the truth model in order to gain computational benefits from implementing an RO Kalman filter. Due to its reduced order, the filter will be capable of handling fast rate measurements.

1. The Reduced-Order Filter Design Model (ROM)

Consider the following reduced-order model

$$x_r(k) = \Phi_r(k,k-1)x_r(k-1) + w_r(k-1) \tag{26}$$

where $\Phi_r(k,k-1)$ is the state transition matrix between two subsequent measurement points at times $t(k-1)$ and $t(k)$, respectively. $\{w_r(k-1)\}$ is an RO random, zero-mean, white noise sequence with the following statistics:

$$E\{w_r(k)\} = 0 \qquad \forall k \geq 0 \tag{27a}$$

$$E\{w_r(k)w_r^T(j)\} = Q_r(k)\delta(k-j) \qquad \forall k,j > 0 \tag{27b}$$

The RO discrete observation model takes the form,

$$z(k) = H_r(k)x_r(k) + v_r(k) \tag{28}$$

where $H_r(k)$ is the RO observation matrix at time $t(k)$ and $\{v_r(k)\}$ is a random, zero-mean, white noise sequence, that is,

$$E\{v_r(k)\} = 0 \qquad \forall k \geq 0 \tag{29a}$$

$$E\{v_r(k)v_r^T(j)\} = R_r(k)\delta(k-j) \qquad \forall k,j \geq 0 \tag{29b}$$

$\{v_r(k)\}$ and $\{w_r(k)\}$ are assumed to be uncorrelated.

The initial state $x_r(0)$ is a random vector which has the following statistics:

$$E\{x_r(0)\}=\mu_r \tag{30a}$$

$$E\{x_r(0)x_r^T(0)\}=P_r(0) \tag{30b}$$

and is uncorrelated with neither $\{w_r(k)\}$ nor $\{v_r(k)\}$, that is,

$$E\{x_r(0)w_r^T(k)\}=0 \qquad \forall k\geq 0 \tag{31a}$$

$$E\{x_r(0)v_r^T(k)\}=0 \qquad \forall k\geq 0 \tag{31b}$$

2. The Reduced-Order Filter Equations

The RO filter generates a linear estimate $\hat{x}_r(k|k)$ of $x_r(k)$ based on the measurement $z(k)$ according to

$$\hat{x}_r(k|k)=\hat{x}_r(k|k-1)+K_r(k)[z(k)-H_r(k)\hat{x}_r(k|k-1)] \tag{32}$$

where the gain matrix $K_r(k)$ may be either predetermined or recursively computed, using the well-known Kalman filter formulation. The projected estimate $\hat{x}_r(k|k-1)$ is calculated in consistence with the ROM as

$$\hat{x}_r(k|k-1)=\Phi_r(k,k-1)\hat{x}_r(k-1|k-1) \qquad \hat{x}_r(0|0)=\mu_r \tag{33}$$

Obviously, errors are introduced into the estimation process since both the process and observation models deviate from the truth model. Therefore, the value of $\hat{x}_r(k|k)$ computed at $k=Np$ and used for renovating the FO filter will corrupt the renovated estimate. This motivates the development of the actual estimation error equations for the combined FO and RO filters. This is presented next.

D. True Estimation Error Equations

The ROM and FOM states which were chosen for the data compression technique analyzed in this work are both products of a linear transformation on x_t, the actual process state. The estimates, $\hat{x}_r$ and $\hat{x}_f$, correspond to their respective models (described in Secs. IV. C.1 and IV. B.1, respectively). Therefore, the respective true estimation errors are:

$$\tilde{x}_r=T_rx_t-\hat{x}_r \tag{34a}$$

$$\tilde{x}_f=T_fx_t-\hat{x}_f \tag{34b}$$

where T_r is the matrix which transforms the true state of the actual model into the ROM state and similarly T_f transforms the true state into the FOM state.

1. RO True Estimation Error

When $x_t(k)$ from Eq. (15) and $\hat{x}_r(k|k-1)$ from Eq. (33) are substituted into Eq. (34a), the following propagation equation of the RO true estimation error is obtained

$$\tilde{x}_r(k|k-1)=\Phi_r(k,k-1)\tilde{x}_r(k-1|k-1)$$
$$+\Delta\Phi_r(k,k-1)x_t(k-1)+T_rw_t(k-1) \tag{35}$$

where

$$\Delta\Phi_r=T_r\Phi_t-\Phi_rT_r \tag{36}$$

Note that $\Delta\Phi_r$ expresses the process dynamics modeling error in the ROM and that $\tilde{x}_r(k|k-1)$ is driven by the actual process noise sequence, $w_t(k-1)$, and not by $w_r(k-1)$. In a similar way, when $\hat{x}_r(k|k)$ from Eq. (32) together with $z(k)$ from Eq. (17) are substituted into Eq. (34a) the following update equation of the RO actual estimation error is obtained:

$$\tilde{x}_r(k|k)=[I-K_r(k)H_r(k)]\tilde{x}_r(k|k-1)$$
$$-K_r(k)\Delta H_r(k)x_t(k)-K_r(k)v_t(k) \tag{37}$$

where

$$\Delta H_r=H_t-H_rT_r \tag{38}$$

Note that ΔH_r expresses the observation modeling error in the ROM and that $\tilde{x}_r(k|k)$ is driven by the actual measurement noise, $v_t(k)$, rather than by $v_r(k)$.

2. FO True Estimation Error

When $x_t(k)$ from Eq. (15) and $\hat{x}_f(k|Np-N)$ from Eq. (25) are substituted into Eq. (34b) the following propagation equation of the FO true estimation error is obtained:

$$\tilde{x}_f(k|Np-N)=\Phi_f(k,k-1)\tilde{x}_f(k-1|Np-N)$$
$$+\Delta\Phi_f(k,k-1)x_t(k-1)+T_fw_t(k-1) \tag{39}$$

where

$$\Delta\Phi_f=T_f\Phi_t-\Phi_fT_f \tag{40}$$

$\Delta\Phi_f$ plays the same role in Eq. (39) as $\Delta\Phi_r$ in Eq. (35). Similarly, $\tilde{x}(k|Np-N)$ is driven by $w_t(k-1)$, the actual process noise sequence, and not by $w_f(k-1)$. Now, using the compact form formulation of the state renovation given in Eq. (7) together with Eqs. (34), the following estimation error equation of the renovated state is obtained,

$$\tilde{x}_f(Np|Np)=[I-K_f(Np)H_f(Np)]\tilde{x}_f(Np|Np-N)$$
$$-K_f(Np)\Delta H_f(Np)x_t(Np)+K_f(Np)\,{}^{\nu}\tilde{x}_r(k|k) \tag{41}$$

where

$$\Delta H_f={}^{\nu}T_r-H_fT_f \tag{42}$$

${}^{\nu}T_r$ consists of the upper ν rows of T_r and ${}^{\nu}\tilde{x}_r(k|k)$ is the true estimation error of the first ν elements in $\tilde{x}_r(k|k)$, given in Eq. (37), at the renovation instant $k=Np$. Note that the FO estimation error depends on the RO estimation error due to the existence of ${}^{\nu}\tilde{x}_r(k|k)$ in Eq. (41).

According to the data compression technique, renovation is followed by state adjustment. To obtain the relationship between $\tilde{x}_r(Np|Np)$, the RO estimation error after adjustment, and $\tilde{x}_f(Np|Np)$, use Eqs. (14a) and (34b) to obtain:

$$\tilde{x}_r(Np|Np)=\Gamma\tilde{x}_f(Np|Np)+\Delta\Gamma x_t(Np) \tag{43}$$

where

$$\Delta\Gamma=T_r-\Gamma T_f \tag{44}$$

From the outline of the data compression technique[5,6] it is clear that the ROM is a subset of the FOM, hence the mapping T_r of x_t into x_r is identical to the mapping ΓT_f of x_t into x_r. Consequently [see Eq. (44)]

$$\Delta\Gamma=0 \tag{45}$$

From Eq. (9) and the same arguments, it is also clear that ΔH_f consists of the first ν rows of $\Delta\Gamma$; therefore [see Eq. (42)]

$$\Delta H_f=0 \tag{46}$$

as well. Note that after adjustment, $\tilde{x}_r(k-1|k-1)$ in Eq. (35) has to be replaced by $\tilde{x}_r(Np|Np)$ for the subsequent propagation step.

E. True Mean Estimation Error Equations

The propagation of the true mean estimation error is obtained by taking the expected value of both sides of Eqs. (35), (37), (39), (41), and (43) as follows:

$$E\{\tilde{x}_r(k|k-1)\} = \Phi_r(k,k-1)E\{\tilde{x}_r(k-1|k-1)\} + \Delta\Phi_r(k,k-1)E\{x_t(k-1)\} \quad (k=1,2\ldots) \tag{47a}$$

$$E\{\tilde{x}_r(k|k)\} = [I-K_r(k)H_r(k)]E\{\tilde{x}_r(k|k-1)\} - K_r(k)\Delta H_r(k)E\{x_t(k)\} \quad (k=1,2\ldots) \tag{47b}$$

$$E\{\tilde{x}_f(k|Np-N)\} = \Phi_f(k,k-1)E\{\tilde{x}_f(k-1|Np-N)\} + \Delta\Phi_f(k,k-1)E\{x_t(k-1)\} \quad (k=1,2\ldots) \tag{47c}$$

$$E\{\tilde{x}_f(Np|Np)\} = [I-K_f(Np)H_f(Np)]E\{\tilde{x}_f(Np|Np-N)\} + K_f(Np)E\{{}^{\nu}\tilde{x}_r(k|k)\} \quad (k=Np) \tag{47d}$$

$$E\{\tilde{x}_r(Np|Np)\} = \Gamma E\{\tilde{x}_f(Np|Np)\} \quad (k=Np) \tag{47e}$$

Note that Eq. (16a) was used in the derivation of Eqs. (47a) and (47c), and Eq. (18a) was used to derive Eq. (47b). Equation (46) was used to derive Eq. (47d) and, similarly, Eq. (45) was used in the derivation of Eq. (47e). It is clear from Eqs. (15) and (19a) that, in general, $E\{x_t\} \neq 0$. Therefore, if a modeling error does exist, that is, if at least $\Delta\Phi_r \neq 0$, $\Delta H_r \neq 0$, or $\Delta\Phi_f \neq 0$, it can be seen from Eqs. (47) that the estimates $\hat{x}_r$ and $\hat{x}_f$ *are biased.*

F. True Covariance Equations

From Eqs. (35), (37), (39), (41), and (43), it is concluded that x_t, $\tilde{x}_r$, and $\tilde{x}_f$ are interrelated. It is, therefore, advantageous to augment these states and propagate them together. This can be done now since the true estimation error equations are successfully expressed in the common form of the error and sensitivity analysis equations.[8,9] Define an augmented state vector y as follows:

$$y^T = [x_t^T, \tilde{x}_r^T, \tilde{x}_f^T] \tag{48}$$

Use Eqs. (35), (37), (39), (41), and (43) to show that y can be propagated according to the following relations.

Fast propagation:

$$y(k|k-1) = \Phi_y(k,k-1)y(k-1|k-1) + G_y w_t(k-1) \quad (k=1,2\ldots) \tag{49a}$$

Fast update:

$$y(k|k) = [I-K_y(k)H_y(k)]y(k|k-1) - K_y(k)v_t(k) \quad (k=1,2\ldots) \tag{49b}$$

Slow renovation and adjustment:

$$y(Np|Np) = \Gamma_y[I-K_y(Np)H_y(Np)]y(k|k) \quad (k=Np) \tag{49c}$$

where

$$\Phi_y = \begin{bmatrix} \Phi_t & 0 & 0 \\ \Delta\Phi_r & \Phi_r & 0 \\ \Delta\Phi_f & 0 & \Phi_f \end{bmatrix} \qquad G_y = \begin{bmatrix} I \\ T_r \\ T_f \end{bmatrix} \tag{50}$$

$$K_y(k) = \begin{bmatrix} 0 \\ K_r(k) \\ 0 \end{bmatrix} \qquad H_y(k) = [\Delta H_r(k) \;\vdots\; H_r(k) \;\vdots\; 0] \tag{51}$$

$$K_y(Np) = \begin{bmatrix} 0 \\ 0 \\ K_f(Np) \end{bmatrix} \qquad H_y(Np) = [0 \;\vdots\; -{}^{\nu\nu}I \;\vdots\; H_f(Np)] \tag{52a}$$

$$\Gamma_y = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & \Gamma \\ 0 & 0 & I \end{bmatrix} \tag{52b}$$

Note that following these definitions of Φ_y and G_y, Eqs. (35) and (39) merged into the single Eq. (49a). This assures the tracking of the correlation between $\tilde{x}_f(Np|Np-N)$ and ${}^{\nu}\tilde{x}_r(k|k)$ which evolves during the fast updating of the RO filter. Also note that Eqs. (41) and (43) merged into the single Eq. (49c).

The existence of the conditions spelled out in Eqs. (45) and (46) are due to the fact that the ROM states are a subset of the FOM states. However, if a more general linear relation between the ROM and FOM states is chosen, then $\Delta\Gamma$ and ΔH_f have to be incorporated into $H_y(Np)$ and Γ_y as follows:

$$H_y(Np) = [\Delta H_f(Np) \;\vdots\; -{}^{\nu\nu}I \;\vdots\; H_f(Np)] \tag{53a}$$

$$\Gamma_y = \begin{bmatrix} I & 0 & 0 \\ \Delta\Gamma & 0 & \Gamma \\ 0 & 0 & I \end{bmatrix} \tag{53b}$$

Define C_y, the second moment matrix of y, as follows:

$$C_y \triangleq E\{yy^T\} \tag{54}$$

Now using Eqs. (49) to compute C_y, the following propagation equations are obtained:

$$C_y(k|k-1) = \Phi_y(k,k-1)C_y(k-1|k-1)\Phi_y^T(k,k-1) + G_y Q_t(k-1)G_y^T \quad (k=1,2,\ldots) \tag{55a}$$

$$C_y(k|k) = [I-K_y(k)H_y(k)]C_y(k|k-1) \cdot [I-K_y(k)H_y(k)]^T + K_y(k)R_t(k)K_y^T(k) \quad (k=1,2,\ldots) \tag{55b}$$

$$C_y(Np|Np) = \Gamma_y[I-K_y(Np)H_y(Np)]C_y(k|k) \cdot [I-K_y(Np)H_y(Np)]^T\Gamma_y^T \quad (k=Np) \tag{56}$$

Note that C_y is computed according to the order outlined in Eqs. (55) at the fast rate at which the measurements are obtained. At $k=Np$ when renovation and adjustment take place, the computation of $C_y(k|k)$ is followed by the computation of $C_y(Np|Np)$ according to Eq. (56). The matrix $C_y(Np|Np)$ replaces $C_y(k-1|k-1)$ in Eq. (55a) for the subsequent computation of $C_y(k|k-1)$.

The second moment matrix C_y is identical to the covariance matrix V_y of y if $E\{y\}=0$. However, it was shown in Sec. IV.E that the latter does not generally hold. In such case, V_y is computed using

$$V_y = C_y - E\{y\}E\{y^T\} \tag{57}$$

and $E\{y\}$ is computed using Eqs. (47) as follows:

$$E\{y(k|k-1)\} = \Phi_y(k,k-1)E\{y(k-1|k-1)\} \qquad (k=1,2,\ldots) \tag{58a}$$

$$E\{y(k|k)\} = [I - K_y(k)H_y(k)]E\{y(k|k-1)\} \qquad (k=1,2,\ldots) \tag{58b}$$

$$E\{y(Np|Np)\} = \Gamma_y[I - K_y(Np)H_y(Np)]E\{y(k|k)\} \qquad (k=Np) \tag{59}$$

The computation of Eqs. (58) and (59) is performed together with Eqs. (55) and (56), respectively.

The inclusion of Eqs. (58) and (59) in the error and sensitivity analysis scheme, when $E\{y\} \neq 0$, is essential to the proper computation of the covariance V_y of the true estimation errors.

Equations (55-59) constitute the error and sensitivity analysis scheme for the new data compression technique.

The covariance matrices of the actual estimation errors is obtained from V_y by partitioning as follows:

$$V_y = \left[\begin{array}{c|c|c} V_{x_t x_t} & V_{x_t \tilde{x}_r} & V_{x_t \tilde{x}_f} \\ \hline V^T_{x_t \tilde{x}_r} & V_{\tilde{x}_r \tilde{x}_r} & V_{\tilde{x}_r \tilde{x}_f} \\ \hline V^T_{x_t \tilde{x}_f} & V^T_{\tilde{x}_r \tilde{x}_f} & V_{\tilde{x}_f \tilde{x}_f} \end{array}\right] \tag{60}$$

where $V_{\tilde{x}_r \tilde{x}_r}$ and $V_{\tilde{x}_f \tilde{x}_f}$ are the covariance matrices of the true estimation errors of the RO and FO filters, respectively.

V. Application of the Error Analysis Scheme to INS

In order to evaluate the performance of the new data compression technique, a computer code that implements the true estimation error mean and covariance equations was written. Then, a typical 27-state INS error model was used as a test case. This model constituted the FOM as well as the truth model of the data compression technique. The 27 states consisted of 3 position, 3 velocity, and 3 attitude states. In addition, the model included one bias and two Markov states for each of the three accelerometers and three gyros of the INS. The nine-state ROM contained only the three position, three velocity, and three attitude errors. The rationale behind this choice is explained in Ref. 6. The observation model consisted of three position measurements with 50, 50, and 10 m rms white-noise measurement errors, respectively. Only the three estimated position errors of the RO filter were used to renovate the FO filter state. Consequently, the pertinent matrices take the following values:

$$\Gamma = [{}^{9\ 9}I \mid {}^{9\ 18}0]$$

where ${}^{jj}I$ is an identity matrix of order j and ${}^{jk}0$ is a $j \times k$ null matrix,

$$H_t = [{}^{3\ 3}I \mid {}^{3\ 24}0] \qquad H_f = [{}^{3\ 3}I \mid {}^{3\ 24}0] \qquad H_r = [{}^{3\ 3}I \mid {}^{3\ 6}0]$$

$$T_f = {}^{27\ 27}I \qquad T_r = [{}^{9\ 9}I \mid {}^{9\ 18}0]$$

$$\Delta H_r = H_t - H_r T_r = {}^{3\ 27}0 \qquad \Delta H_f = {}^{3}T_r - H_f T_f = {}^{3\ 27}0$$

$$\Delta\Gamma = T_r - \Gamma T_f = {}^{9\ 27}0$$

The INS was flown in a straight and level trajectory for about 8 minutes to simulate a benign in-flight alignment phase. Four different data processing cases were considered and presented: 1) a Kalman update of the FO filter every second; 2) a Kalman update of the FO filter every 16 seconds, discarding the intermediate measurements; 3) using a 16/1 data compression ratio in the analyzed data compression technique, a Kalman update of the RO filter every second and a renovation of the FO filter estimate every 16 seconds; 4) a Kalman update of the RO filter alone every second.

Figure 2 presents the true east position estimation error behavior for these four cases. Although the position is measured directly, the RO filter true estimate without renovation followed by adjustment (case 4) diverges. However, when the steps of the new technique are followed in full (case 3), the RO filter true estimate practically coincides with the optimal estimate of case 1 which has the minimum attainable error at this processing rate. Figure 3 presents the true east velocity estimation error behavior. Here too it is evident that the true estimate diverges when the RO filter is

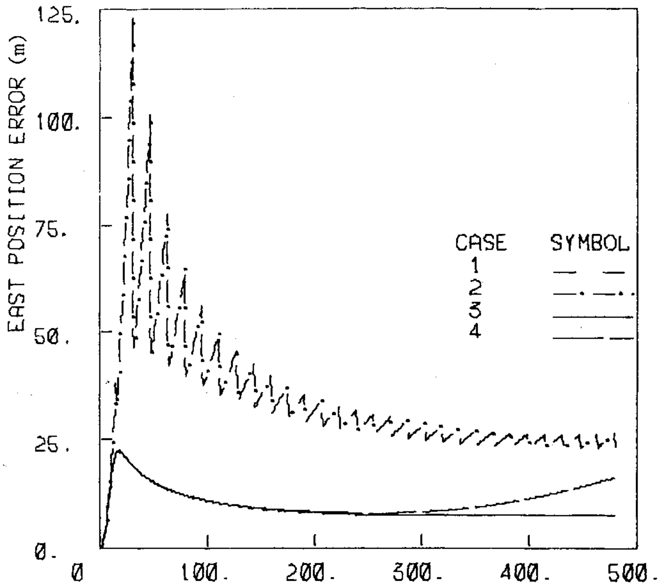

Fig. 2 Standard deviation of the true east position estimation error.

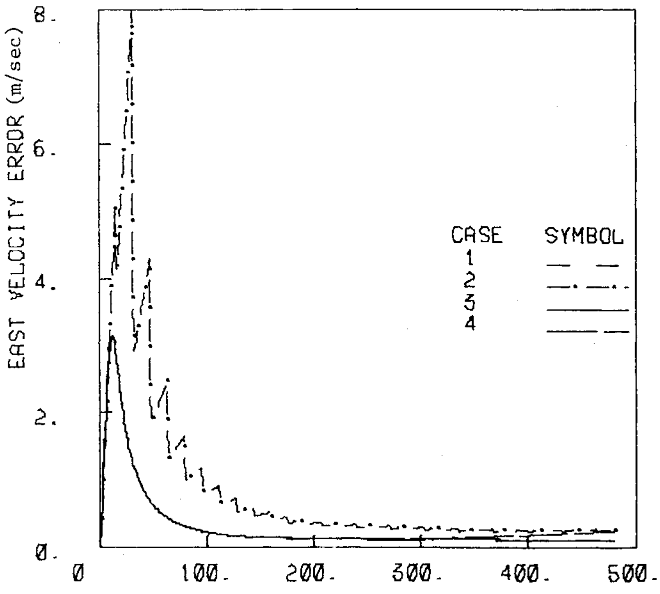

Fig. 3 Standard deviation of the true east velocity estimation error.

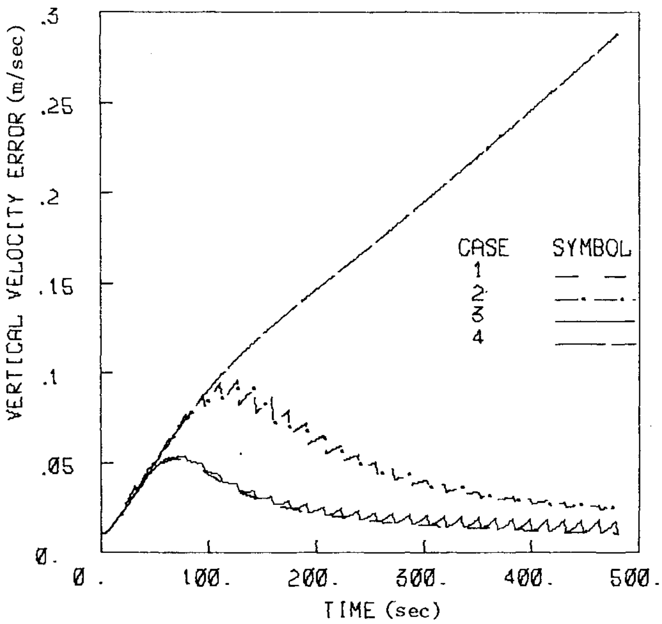

Fig. 4 Standard deviation of the true vertical velocity estimation error.

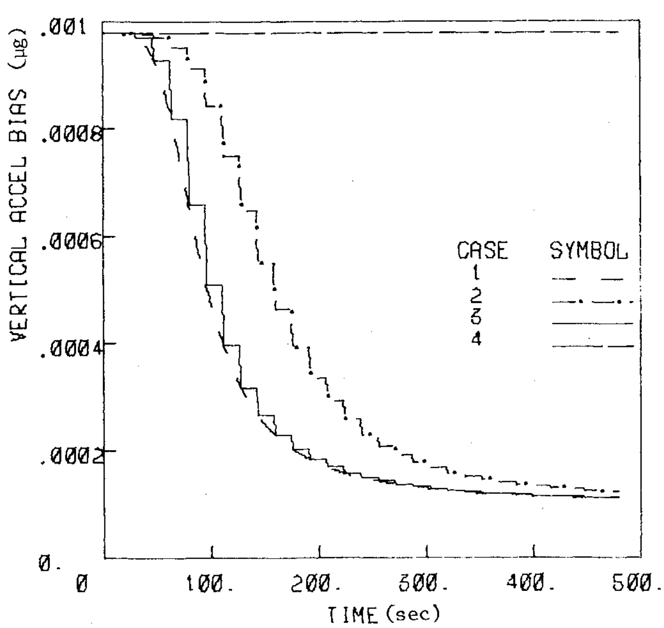

Fig. 5 Standard deviation of the true vertical accelerometer-bias estimation error.

mechanized alone. However, if the new technique is utilized in full, the RO filter true estimate practically coincides with the optimal one (case 1) although velocity is not measured directly.

Figure 4 presents the true vertical velocity estimation error behavior. Note the fast divergence of the true estimation error of case 4. Also note the effect of the adjustment step in case 3 which brings the true estimation error down to the optimal estimate of case 1. These results indicate that the ROM describes very well the behavior of the first nine states of the truth model in short intervals (of 16 s). Following renovation, their values in the RO filter are replaced by their corresponding renovated values from the FO filter (adjustment). Consequently, the ROM yields a good description of the truth model for longer time intervals. It is evident that the adjustment step is necessary in order to prevent divergence.

The true vertical accelerometer bias estimation error behavior is presented in Fig. 5. Note that even though it is not modeled at all in the RO filter, its true renovated estimate closely approaches the optimal value (of case 1). Using the RO filter alone (case 4) its value is not estimated at all. Although only the first three position states of the RO filter were used to renovate the FO filter, it is evident from these figures that the information content of the fast rate measurements was fully acquired by the FO filter through the slow rate renovation process. It is interesting to note in Fig. 5 that after a long enough time the estimation error of all first three cases reaches the same value. The error obtained using the data compression technique is very close to the optimal case anyway, but even the approach of case 2 to the other two cases is not surprising since in case 2 the information lost, when discarding intermediate data, is gained when enough information is added to the filter after enough time passes. This does not happen, however, with position (Fig. 2) and velocity (Figs. 3 and 4) estimation errors, due to their divergent nature in the 16 s intervals.

These results and particularly the one presented in Fig. 5, clearly demonstrate the ability of the new data compression technique to produce accurate estimates of the full state.

VI. Conclusions

True error equations were successfully developed for the new data compression technique. This was accomplished after reformulating the equations of the data compression algorithm in a compact form. The resulting true error mean and covariance propagation equations constitute the error and sensitivity analysis scheme of the new data compression technique. These equations served as a tool for evaluating the performance of the suggested algorithm when applied to an inertial navigation system model. The results demonstrate that, although the algorithm is a suboptimal one, the true estimation errors are very close to the optimal estimation errors which were obtained through the application of a full-order Kalman filter update at every measurement. Consequently, the proposed new data compression technique is a viable candidate in cases where data compression methods are considered.

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